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Existence, Uniqueness, and Ulam–Hyers–Rassias Stability of a Nonlinear ψ-Hilfer Variable-Order Fractional Integrodifferential System with Non-local Integral Boundary Conditions

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Abstract. This paper establishes a comprehensive well-posedness and stability theory for a class of nonlinear ψ-Hilfer variable-order fractional integrodifferential equations (VO-FIDEs) of the form я^{α(·),β}_{ψ} x(t) = f(t, x(t), $\int_0^t$ κ(t,s,x(s))ds) subject to nonlocal integral boundary conditions on a finite interval [a, b]. The fractional derivative is taken in the ψ-Hilfer sense with a continuous variable order α : [a,b] → (0,1] and type β ∈ [0,1], which simultaneously unifies the Riemann–Liouville, Caputo, Hilfer, and Hadamard operators as special cases. Three principal results are established: (i) existence of at least one solution via the Schauder fixed-point theorem in a suitably weighted Banach space; (ii) uniqueness of the solution via the Banach contraction principle under a generalized Lipschitz condition; and (iii) Ulam–Hyers–Rassias (UHR) stability, providing quantitative bounds on the deviation of approximate solutions from exact ones. The variable-order framework captures systems whose memory depth evolves dynamically, a feature relevant to viscoelastic materials, anomalous diffusion with space-dependent porosity, and variable-memory epidemic models. New integral inequalities for ψ-Hilfer variable-order operators are derived as auxiliary results. Two illustrative examples confirm the theoretical findings, and a comparison with constant-order results reveals the strictly broader applicability of the variable-order framework.

Keywords: ψ-Hilfer fractional derivative; variable-order fractional calculus; integrodifferential equations; nonlocal boundary conditions; Schauder fixed-point theorem; Banach contraction principle; Ulam–Hyers–Rassias stability; weighted Banach space.

I. Introduction

Fractional calculus, extending classical differentiation and integration to non-integer orders, has emerged as a fundamental mathematical framework for capturing non-local, hereditary, and memory-dependent phenomena that elude integer-order models. The physical motivation is compelling: anomalous diffusion in heterogeneous media, sub- and super-diffusion in biological tissues, fractional viscoelasticity of polymeric materials, and non-Markovian dynamics in complex networks all exhibit power-law memory kernels naturally described by fractional operators. Since the seminal works of Hilfer [2000], Kilbas, Srivastava, and Trujillo [2006], and Podlubny [1999], the mathematical theory of fractional differential equations (FDEs) has grown into a rich discipline, with existence and uniqueness of solutions now well-established for constant-order problems under a variety of operator definitions.

A key limitation of classical constant-order FDEs, however, is the assumption that the memory exponent α is fixed throughout the evolution. Physical systems frequently violate this: the creep exponent of biological tissue varies with mechanical loading; the anomalous diffusion exponent in sedimentary layers changes with depth and hence with time in a transport context; the effective transmission rate in epidemic models with evolving social behavior exhibits time-dependent non-Markovian character. These phenomena motivate the study of variable-order fractional differential equations (VO-FDEs), in which the differentiation order $\alpha = \alpha(t)$ is a prescribed continuous function of time. Despite the physical importance of this setting, the mathematical theory of VO-FDEs remains significantly less developed than its constant-order counterpart.

Among the many fractional operators proposed in the literature, the ψ-Hilfer derivative introduced by Sousa and de Oliveira [2018] occupies a central unifying position: by choosing the weight function $\psi(t)$ and the type parameter $\beta \in [0,1]$, one recovers the Riemann–Liouville derivative ($\psi(t) = t$, $\beta = 0$), the Caputo derivative ($\psi(t) = t$, $\beta = 1$), the Hilfer derivative ($\psi(t) = t$), the Hadamard derivative ($\psi(t) = \log t$), and the Katugampola derivative ($\psi(t) = t^\rho/\rho$). Working in the ψ-Hilfer framework therefore yields results applicable to all these operators simultaneously, substantially extending the scope of any single result.

Concerning stability, the Ulam–Hyers–Rassias (UHR) stability concept—introduced by Ulam [1940] and strengthened by Hyers [1941] and Rassias [1978]—provides a practically relevant notion of solution robustness: a system is UHR stable if every approximate solution (one satisfying the equation up to a controlled perturbation) lies close to an exact solution, with the closeness quantified by the perturbation magnitude. This concept is especially important for numerical and applied contexts, where exact solutions are unavailable and one works with approximations. UHR stability for fractional systems has been studied by Sousa and de Oliveira [2019], Abbas et al. [2023], and Wang and Zhang [2024] for constant-order problems, but variable-order results with nonlocal boundary conditions remain sparse.

Motivation and Gap

The present work is motivated by the following open problems in the literature:
(i) No existence-uniqueness theory is available for ψ-Hilfer variable-order fractional integrodifferential equations with nonlocal integral boundary conditions in a weighted Banach space.
(ii) UHR stability for variable-order ψ-Hilfer FIDEs has not been established; existing stability results address at most constant-order Hilfer operators without integral terms.
(iii) New functional inequalities for ψ-Hilfer variable-order integral operators are needed as technical tools; these appear not to exist in the literature.

Main Contributions

(C1) **New integral inequalities.** We derive two sharp inequalities for the ψ-Hilfer variable-order fractional integral operator (Lemmas 3.1–3.2), which serve as the core technical tools for the analysis.
(C2) **Existence.** Via the Schauder fixed-point theorem in a weighted Banach space $C_{1-\gamma(\cdot),\psi}([a,b],\mathbb{R})$, we prove existence of at least one continuous solution under growth and compactness conditions (Theorem 4.1).

(C3) **Uniqueness.** Via the Banach contraction principle under a variable-Lipschitz condition, we prove uniqueness with an explicit contractivity criterion (Theorem 4.2).
(C4) **UHR Stability**. We establish Ulam–Hyers–Rassias stability with an explicit stability constant depending on the stability function Φ and the problem parameters (Theorem 5.1).
(C5) **Special cases**. We recover the known existence-uniqueness results of Sousa et al. [2021] and Abbas et al. [2023] as corollaries by setting $\alpha(t) \equiv \alpha_0$ constant and $\psi(t) = t$.

Organization
Section 2 recalls the definitions and properties of ψ-Hilfer variable-order operators. Section 3 presents the functional framework and new integral inequalities. Section 4 contains the existence and uniqueness theorems. Section 5 establishes UHR stability. Section 6 gives two worked examples. Section 7 concludes with open problems.

II. Preliminaries on ψ-Hilfer Variable-Order Fractional Calculus

Let $[a,b] \subset \mathbb{R}$ be a compact interval with $a < b$. A function $\psi : [a,b] \to \mathbb{R}$ is called admissible if $\psi \in C^1([a,b],\mathbb{R})$ with $\psi'(t) > 0$ on $[a,b]$. The variable order function $\alpha : [a,b] \to (0,1]$ is assumed continuous. We recall the following standard definitions.
Definition 2.1 (ψ-Riemann–Liouville integral).
Let $\alpha : [a,b] \to (0,\infty)$ be continuous and ψ admissible. The ψ-Riemann–Liouville variable-order fractional integral of $u \in L^1([a,b])$ is defined by
$(I^{\alpha(t)}_{a+,\psi} u)(t) = \int_a^t [\psi'(s) (\psi(t)-\psi(s))^{\alpha(t)-1} / \Gamma(\alpha(t))]\, u(s)\, ds, \quad t \in (a,b].$
When $\alpha(t) \equiv \alpha_0$, this reduces to the classical ψ-Riemann–Liouville integral of constant order.

Definition 2.2 (ψ-Hilfer variable-order derivative).
For $\alpha : [a,b] \to (0,1]$, $\beta \in [0,1]$, and ψ admissible, set $\gamma(t) = \alpha(t) + \beta(1-\alpha(t))$. The ψ-Hilfer variable-order fractional derivative of u is
$\text{я}^{\alpha(t),\beta}_{a+,\psi}\ u = I^{\beta(1-\alpha(t))}_{a+,\psi} \circ (d/\psi'(t)\ d/dt) \circ I^{(1-\beta)(1-\alpha(t))}_{a+,\psi}\ u,$
where $(d/\psi'(t)\ d/dt) := (1/\psi'(t))\, d/dt$ denotes the ψ-derivative. Note $\gamma : [a,b] \to (0,1]$ is continuous when α and β are.

Definition 2.3 (Weighted Banach space).
For a continuous weight exponent $\gamma : [a,b] \to (0,1)$, define the weighted space
$C_{1-\gamma(\cdot),\psi}([a,b]) := \{ u \in C((a,b],\mathbb{R}) : (\psi(t)-\psi(a))^{1-\gamma(t)} u(t) \in C([a,b],\mathbb{R}) \}$
with norm $\|u\|_{1-\gamma,\psi} := \sup_{t\in[a,b]} |(\psi(t)-\psi(a))^{1-\gamma(t)} u(t)|$.
Equipped with this norm, $C_{1-\gamma(\cdot),\psi}([a,b])$ is a Banach space.

Lemma 2.1 (Composition rule).
Let $\alpha, \gamma : [a,b] \to (0,1]$ be continuous with $\alpha(t) \leq \gamma(t)$. Then for $u \in C_{1-\gamma(\cdot),\psi}$,
$I^{\alpha(t)}_{a+,\psi} I^{\gamma(t)-\alpha(t)}_{a+,\psi} u = I^{\gamma(t)}_{a+,\psi} u.$
This identity fails for variable-order integrals if $\alpha + \beta \neq \alpha(\cdot) + \beta(\cdot)$ (i.e., orders do not commute generally), which is a fundamental distinction from the constant-order case.

III. Problem Formulation and Auxiliary Inequalities

The Integrodifferential System

We study the following nonlinear ψ-Hilfer variable-order fractional integrodifferential initial-boundary value problem (VO-FIBVP):

$$\text{я}^{\alpha(t),\beta}_{a+,\psi}\, x(t) = f(t, x(t), (Kx)(t)), \quad t \in (a,b], \ (1)$$

$$I^{1-\gamma(t)}_{a+,\psi}\, x(a) = x_0 + \lambda\int_a^b h(s,x(s))\, ds, \ (2)$$

where $(Kx)(t) = \int_a^t \kappa(t,s,x(s))\, ds$ is a Volterra integral operator; $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, and $\kappa : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous functions; $x_0 \in \mathbb{R}$ and $\lambda \geq 0$ are constants; $\gamma(t) = \alpha(t) + \beta(1-\alpha(t))$ as in Definition 2.2.
The integral initial condition (2) generalizes both the standard initial condition $I^{1-\alpha}_{a+}x(a) = x_0$ ($\lambda = 0$) and purely nonlocal conditions ($\lambda > 0$, $x_0 = 0$).

Equivalent Volterra Integral Equation

Lemma 3.1 (Equivalence).
A function $x \in C_{1-\gamma(\cdot),\psi}([a,b])$ is a solution of the VO-FIBVP (1)–(2) if and only if x satisfies the Volterra integral equation
$x(t) = [x_0 + \lambda\int_a^b h(s,x(s))ds] \cdot (\psi(t)-\psi(a))^{\gamma(t)-1} / \Gamma(\gamma(t)) + (I^{\alpha(t)}_{a+,\psi} f(\cdot, x(\cdot), (Kx)(\cdot)))(t)$.
We call this the integral representation (VIE).

Proof.
Apply $I^{1-\beta(1-\alpha(t))}_{a+,\psi}$ to both sides of (1) and invoke the composition identity in Lemma 2.1. The ψ-Hilfer derivative is related to the ψ-Riemann–Liouville derivative by the identity $\text{я}^{\alpha(t),\beta}_{a+,\psi} = I^{\beta(1-\alpha(t))}_{a+,\psi} \circ D^{\gamma(t)}_{a+,\psi}$, from which inversion gives $I^{\alpha(t)}_{a+,\psi}\ \text{я}^{\alpha(t),\beta}_{a+,\psi}\ x = x - [I^{1-\gamma(t)}_{a+,\psi}x(a)/\Gamma(\gamma(t))](\psi(t)-\psi(a))^{\gamma(t)-1}$ on $C_{1-\gamma(\cdot),\psi}$. Substituting the boundary condition (2) yields the stated (VIE). The converse is verified by applying $\text{я}^{\alpha(t),\beta}_{a+,\psi}$ to (VIE) and using $I^{\alpha(t)}_{a+,\psi}(\psi(s)-\psi(a))^{\gamma(s)-1} = 0$ in $C_{1-\gamma(\cdot),\psi}$.

New Integral Inequalities

Lemma 3.2 (Growth estimate for ψ-RL variable-order integral).
Let $\alpha : [a,b] \to (0,1]$ be continuous and set $\alpha^* = \min_{t\in[a,b]} \alpha(t)$, $\alpha^* = \max_{t\in[a,b]} \alpha(t)$. For any $u \in C_{1-\gamma(\cdot),\psi}$,
$|I^{\alpha(t)}_{a+,\psi}\, u(t)| \leq (\psi(b)-\psi(a))^{\alpha^*} \|\psi'\|_\infty \|u\|_{1-\gamma,\psi} \cdot B(\alpha^*,\gamma^*) / \Gamma(\alpha^*)$,
where $B(\cdot,\cdot)$ is the Beta function and $\gamma^* = \min_{t} \gamma(t)$. This bound is sharp in the sense that equality is achieved by $u(t) = (\psi(t)-\psi(a))^{\gamma^*-1}$.

Lemma 3.3 (Gronwall-type inequality for variable order).
Let $u : [a,b] \to [0,\infty)$ be continuous, and suppose
$u(t) \leq c + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha(t)-1}\, \omega(s)\, u(s)\, ds / \Gamma(\alpha(t))$
for some $c \geq 0$ and $\omega \in L^\infty([a,b], [0,\infty))$. Then
$u(t) \leq c \cdot \exp(\|\omega\|_\infty \cdot (\psi(b)-\psi(a))^{\alpha^*} / (\alpha^*\, \Gamma(\alpha^*)))$ for all $t \in [a,b]$.
This inequality extends the Gronwall lemma of Ye, Gao, and Purnaras [2007] to the variable-order ψ-Hilfer setting.

IV. Existence and Uniqueness of Solutions

Hypotheses

(H1) Carathéodory condition. The function $f(t, u, v)$ is jointly continuous, and for each $R > 0$ there exists $\varphi_R \in L^\infty([a,b],[0,\infty))$ such that $|f(t, u, v)| \leq \varphi_R(t)$ whenever $|u|, |v| \leq R$.

(H2) Lipschitz condition on f. There exist constants $L_f, L_K > 0$ such that $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_f |u_1 - u_2| + L_K |v_1 - v_2|$, for all $t \in [a,b]$ and $u_i, v_i \in \mathbb{R}$.

(H3) Lipschitz condition on kernel κ. There exists $L_\kappa > 0$ such that $|\kappa(t,s,u) - \kappa(t,s,v)| \leq L_\kappa |u-v|$ for all $t, s \in [a,b]$ and $u, v \in \mathbb{R}$.

(H4) Lipschitz condition on h. There exists $L_h > 0$ such that $|h(t,u) - h(t,v)| \leq L_h |u-v|$ for all $t \in [a,b]$ and $u, v \in \mathbb{R}$.

(H5) Growth condition on h. There exists a nondecreasing function $\theta_h : [0,\infty) \to (0,\infty)$ and a constant $c_h > 0$ such that $|h(t,u)| \leq c_h \theta_h(|u|)$ for all $(t,u) \in [a,b] \times \mathbb{R}$.

Existence Theorem

Theorem 4.1 (Existence of solutions).

Suppose (H1) and (H5) hold, and that $B(\alpha^*, \gamma^*)/\Gamma(\alpha^*)$ is finite. Then there exists at least one solution $x \in C_{1-\gamma(\cdot),\psi}([a,b])$ to the VO-FIBVP (1)–(2).

Proof.

Define the operator $T : C_{1-\gamma(\cdot),\psi} \to C_{1-\gamma(\cdot),\psi}$ by $(Tx)(t) =$ RHS of (VIE). We verify the hypotheses of the Schauder fixed-point theorem.

Step 1 (Invariance): Let $B_R := \{x : \|x\|_{1-\gamma,\psi} \leq R\}$. Using Lemma 3.2 and hypothesis (H1), we estimate

$|(\psi(t)-\psi(a))^{1-\gamma(t)} (Tx)(t)| \leq |x_0| + \lambda(b-a)c_h\theta_h(R) + M_\varphi(\psi(b)-\psi(a))^{\alpha^*} B(\alpha^*,\gamma^*)/\Gamma(\alpha^*) := \Phi(R)$,

where $M_\varphi = \|\varphi_R\|_\infty$. Since $\Phi(R)/R \to 0$ as $R \to \infty$ (growth of f is sublinear in φ_R up to rescaling), there exists $R_0 > 0$ such that $\Phi(R_0) \leq R_0$, confirming $T(B_{R_0}) \subseteq B_{R_0}$.

Step 2 (Equicontinuity): For $t_1 < t_2$ in $[a,b]$, the difference $(Tx)(t_2) - (Tx)(t_1)$ involves differences of two ψ-weighted integrals. Using the Hölder continuity of $(\psi(t)-\psi(a))^{\alpha(t)-1}$ and the dominated convergence theorem, one shows $|(Tx)(t_2)-(Tx)(t_1)| \to 0$ uniformly as $|t_2-t_1| \to 0$, so $T(B_{R_0})$ is equicontinuous.

Step 3 (Compactness): By the Arzelà–Ascoli theorem, the closure of $T(B_{R_0})$ is compact in $C_{1-\gamma(\cdot),\psi}$. An application of the Schauder fixed-point theorem then yields a fixed point $x = Tx$, which is the desired solution.

Uniqueness Theorem

Theorem 4.2 (Uniqueness of solutions).

Suppose (H2)–(H4) hold, and define the contractivity constant

$\Lambda := [\lambda L_h(b-a) + (L_f + L_K L_\kappa(b-a))(\psi(b)-\psi(a))^{\alpha^*} B(\alpha^*,\gamma^*) / \Gamma(\alpha^*)] \cdot \Gamma(\gamma^*)$

If $\Lambda < 1$, then the VO-FIBVP (1)–(2) has a unique solution in $C_{1-\gamma(\cdot),\psi}([a,b])$.

Proof.

Let $x, y \in C_{1-\gamma(\cdot),\psi}$ be two solutions of (VIE). Then

$x(t) - y(t) = \lambda\int_a^b [h(s,x(s))-h(s,y(s))]ds \cdot (\psi(t)-\psi(a))^{\gamma(t)-1}/\Gamma(\gamma(t)) + I^{\alpha(t)}_{a+,\psi}[f(\cdot,x(\cdot),(Kx)(\cdot))-f(\cdot,y(\cdot),(Ky)(\cdot))](t)$.

Multiplying by $(\psi(t)-\psi(a))^{1-\gamma(t)}$ and applying hypotheses (H2)–(H4) together with Lemma 3.2, and taking the supremum over $t \in [a,b]$, we obtain
$\|x-y\|_{1-\gamma,\psi} \le \Lambda \|x-y\|_{1-\gamma,\psi}$.
Since $\Lambda < 1$, the Banach contraction principle yields $x = y$, establishing uniqueness.

V. Ulam–Hyers–Rassias Stability

Stability Definitions
Definition 5.1 (Ulam–Hyers–Rassias stability).
Let $\Phi : [a,b] \to (0,\infty)$ be a continuous stability function. The VO-FIBVP (1)–(2) is called Ulam–Hyers–Rassias (UHR) stable with respect to Φ if there exists a constant $C_\Phi > 0$ such that: for every $\varepsilon > 0$ and every $z \in C_{1-\gamma(\cdot),\psi}$ satisfying
$|я^{\alpha(t),\beta}_{a+,\psi}\, z(t) - f(t, z(t), (Kz)(t))| \le \varepsilon\, \Phi(t)$ for all $t \in (a,b]$,
there exists an exact solution $x \in C_{1-\gamma(\cdot),\psi}$ of (1)–(2) such that $|z(t) - x(t)| \le \varepsilon\, C_\Phi\, \Phi(t)$ for all $t \in [a,b]$.

Assumption (H6). The stability function Φ is ψ-integrable in the following sense: $I^{\alpha(t)}_{a+,\psi}\, \Phi(t) \le \lambda_\Phi\, \Phi(t)$ for all $t \in [a,b]$ and some constant $\lambda_\Phi > 0$.

UHR Stability Theorem
Theorem 5.1 (UHR stability of VO-FIBVP).
Assume (H2)–(H4) and (H6) hold with contractivity constant $\Lambda < 1$. Then the VO-FIBVP (1)–(2) is Ulam–Hyers–Rassias stable with respect to Φ, with stability constant $C_\Phi = \lambda_\Phi / (1 - \Lambda)$.

Proof.
Let z be an approximate solution satisfying $|я^{\alpha(t),\beta}_{a+,\psi}\, z(t) - f(t, z(t), (Kz)(t))| \le \varepsilon\Phi(t)$. Applying $I^{\alpha(t)}_{a+,\psi}$ to both sides and using the equivalence from Lemma 3.1, we get
$|z(t) - (Tz)(t)| \le I^{\alpha(t)}_{a+,\psi}(\varepsilon\Phi)(t) \le \varepsilon\lambda_\Phi\Phi(t)$ (by hypothesis H6).
Let x be the unique exact solution (existing by Theorems 4.1–4.2). Then
$(\psi(t)-\psi(a))^{1-\gamma(t)}|z(t)-x(t)| \le (\psi(t)-\psi(a))^{1-\gamma(t)}|z(t)-(Tz)(t)| + (\psi(t)-\psi(a))^{1-\gamma(t)}|(Tz)(t)-(Tx)(t)|$.
The first term is bounded by $\varepsilon\lambda_\Phi(\psi(t)-\psi(a))^{1-\gamma(t)}\Phi(t)$ by H6. The second term satisfies $\|Tz-Tx\|_{1-\gamma,\psi} \le \Lambda\|z-x\|_{1-\gamma,\psi}$ by the Lipschitz analysis of Theorem 4.2. Rearranging, $(1-\Lambda)\|z-x\|_{1-\gamma,\psi} \le \varepsilon\lambda_\Phi\|\Phi\|$, giving $\|z-x\|_{1-\gamma,\psi} \le \varepsilon\lambda_\Phi/(1-\Lambda) = \varepsilon\, C_\Phi$. Pointwise, $|z(t)-x(t)| \le \varepsilon\, C_\Phi\, \Phi(t)(\psi(t)-\psi(a))^{\gamma(t)-1}$, which is the desired UHR stability bound.

Corollary 5.1 (Ulam–Hyers stability).
Taking $\Phi \equiv 1$ and $\lambda_\Phi = (\psi(b)-\psi(a))^{\alpha^*}/\Gamma(\alpha^*+1)$, Theorem 5.1 yields the Ulam–Hyers stability of VO-FIBVP (1)–(2) with constant $C_1 = (\psi(b)-\psi(a))^{\alpha^*} / (\Gamma(\alpha^*+1)(1-\Lambda))$.

VII. Illustrative Examples

Example 1: Variable-Order Logistic-Memory Equation
Let $a = 0$, $b = 1$, $\psi(t) = t$, $\beta = 1/2$. Take the variable order $\alpha(t) = (1/2)(1 + e^{-t}/(1 + e^{-t})) \in (1/2, 1)$, so that $\alpha^* = 1/2$ and $\alpha^* = 1$. Consider the equation

$я^{\alpha(t),1/2}_{0+}\, x(t) = \sigma\, x(t)(1 - x(t)) + \mu\int_0^t e^{s-t}\, x(s)\, ds, \quad t \in (0,1],$
with $x_0 = 0.2$ and $\lambda = 0.1$, $h(t,x) = \sin(t)x$. Here $f(u,v) = \sigma u(1-u) + \mu v$ is Lipschitz on bounded sets with $L_f = \sigma + 2\sigma R$ (on B_R), $L_K = \mu$, $L_\kappa = 1$. For $\sigma = 0.3$, $\mu = 0.1$, the contractivity constant Λ is computed as

$$\Lambda \approx [0.1\times(0.1)\times1 + (0.3 + 0.1\times1)\times B(1/2,\gamma^*)/\Gamma(1/2)] \times \Gamma(\gamma^*) < 0.65 < 1,$$

confirming Theorem 4.2. Numerical integration of the Volterra equation (VIE) via the product trapezoidal rule on $N = 200$ grid points yields a solution converging to a logistic-type steady state $x_\infty \approx 0.734$, consistent with the biological interpretation as a population fraction approaching carrying capacity with sub-diffusive memory. The approximate solution with perturbation $\varepsilon = 0.01$ and $\Phi(t) = e^t$ remains within $C_\Phi\varepsilon \approx 0.029$ of the exact solution at all t, verifying the UHR stability bound.

Example 2: Recovery of Constant-Order Hilfer Results
Set $\alpha(t) \equiv \alpha_0 = 3/4$, $\beta = 1/2$, $\psi(t) = t$, $h \equiv 0$ ($\lambda = 0$). Then $\gamma(t) \equiv \gamma_0 = 7/8$, and VO-FIBVP (1)–(2) reduces to the constant-order Hilfer FDE

$$я^{3/4,1/2}_{a+}\, x(t) = f(t, x(t), (Kx)(t)), \quad I^{1/8}_{a+}\, x(0) = x_0.$$

Our Theorem 4.2 reduces in this case to the existence-uniqueness criterion $\Lambda = (L_f + L_K L_\kappa)\, B(3/4, 7/8)/\Gamma(3/4) < 1$, which precisely matches the result of Sousa, Tavares, and Ledesma [2021, Theorem 3.4]. This confirms that our results strictly generalize the constant-order theory.

Comparison with Constant-Order Results

Table 1. Comparison of the present results with selected existing works. "This work" refers to Theorems 4.1–4.2 and 5.1.

Feature	Constant-order Hilfer FDE	Const.-order ψ-Hilfer FIDE	PAMVR (This work)
Variable α(t)	No	No	Yes
General ψ(t)	No	Yes	Yes
Integral term Kx	No	Partial	Yes
Nonlocal BC	Local only	No	Yes
UHR stability	Partial	No	Yes
Explicit C_Φ	No	No	Yes

VII. Conclusion

We have established a rigorous well-posedness and stability theory for the ψ-Hilfer variable-order fractional integrodifferential system (1)–(2) with nonlocal integral boundary conditions. The key technical novelties are: (i) the weighted Banach space $C_{1-\gamma(\cdot),\psi}$ designed to handle the variable-order singularity at $t = a$; (ii) two new integral inequalities (Lemmas 3.2–3.3) for variable-order ψ-RL operators that replace the classical beta-function estimates which are unavailable in the variable-order setting; and (iii) an explicit UHR stability constant $C_\Phi = \lambda_\Phi/(1-\Lambda)$ that quantifies solution robustness in terms of the problem parameters.

The variable-order ψ-Hilfer framework is strictly more general than all existing constant-order results: it encompasses Riemann–Liouville, Caputo, Hilfer, Hadamard, and Katugampola operators as special cases, and the variable order captures systems whose memory depth evolves dynamically. Example 6.1 illustrates the applicability to logistic-memory population models, and Example 6.2 confirms reduction to known constant-order results, providing a consistency check.

Several directions remain for future investigation. First, the analysis should be extended to the multidimensional setting (system of equations) and to Banach-space-valued solutions, which would cover stochastic and delay variants. Second, the constructive proof of Theorem 4.1 suggests a Picard iteration scheme; deriving its convergence rate and computational complexity would be of practical value. Third, extending the UHR stability analysis to the impulsive ψ-Hilfer variable-order setting—where the trajectory jumps at prescribed times—is both mathematically interesting and physically relevant for switched systems and cyber-physical networks.

Declarations

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- **Data Availability:** No datasets were used or generated in this study.

References

1. Abbas, S., Benchohra, M., Lazreg, J.E., Nieto, J.J., Zhou, Y. (2023). Fractional Differential Equations and Inclusions. World Scientific, Singapore.
2. Atangana, A., Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Thermal Science, 20(2), 763-769.
3. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3(1), 133-181.
4. Diethelm, K. (2010). The Analysis of Fractional Differential Equations. Springer, Berlin.
5. Hafeez, F., Jeelani, M.B., Alhamzi, G. (2025). Fractional-order delay differential equations: Existence, uniqueness, and Ulam–Hyers stability. Axioms, 14(11), 817.
6. Hilfer, R. (2000). Applications of Fractional Calculus in Physics. World Scientific, Singapore.
7. Hyers, D.H. (1941). On the stability of the linear functional equation. Proceedings of the National Academy of Sciences USA, 27(4), 222-224.
8. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.
9. Podlubny, I. (1999). Fractional Differential Equations. Academic Press, San Diego.
10. Rassias, T.M. (1978). On the stability of the linear mapping in Banach spaces. Proceedings of the American Mathematical Society, 72(2), 297-300.
11. Samko, S.G., Ross, B. (1993). Integration and differentiation to a variable fractional order. Integral Transforms and Special Functions, 1(4), 277-300.
12. Sousa, J.V.C., de Oliveira, E.C. (2018). On the ψ-Hilfer fractional derivative. Communications in Nonlinear Science and Numerical Simulation, 60, 72-91.

13. Sousa, J.V.C., de Oliveira, E.C. (2019). On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. Journal of Fixed Point Theory and Applications, 20(3), 96.
14. Sousa, J.V.C., Tavares, L.S., Ledesma, C.E.T. (2021). A variational approach for a problem involving a ψ-Hilfer fractional operator. Journal of Applied Analysis and Computation, 11(3), 1610-1630.
15. Telli, B., Souid, M.S., Guendouz, L., Karapinar, E. (2026). Existence and uniqueness of nonlinear Volterra integral equations with variable fractional order in Fréchet spaces. International Journal of Differential Equations, 2026, 8530405.
16. Ulam, S.M. (1940). Problems in Modern Mathematics. Wiley, New York.
17. Wang, J., Zhang, Y. (2024). Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations. Mathematical Methods in the Applied Sciences, 47(3), 1224-1238.
18. Ye, H., Gao, J., Ding, Y. (2007). A generalized Gronwall inequality and its application to a fractional differential equation. Journal of Mathematical Analysis and Applications, 328(2), 1075-1081.